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COMMENT

Remarks on duality, lattices and the Sierpinski gasket

H Nencka-Ficek

Institute of Molecular Physics, Polish Academy of Sciences, Poznań, Smoluchowskiego 17/19, Poland

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Abstract. Duality constructions for the Sierpinski gasket and other fractals are studied. Coxeter's definition of a dual lattice is used to prove the existence of two kinds of dual lattice for the Sierpinski gasket.

An advantage of using the dual transformation as a tool for investigating the phase structure and nature of a variety of systems in quantum field theory and statistical mechanics is unquestionable (Kramers and Wannier 1941, Onsager 1944, Wegner 1971, Kadanoff and Ceva 1971, Feynman 1972, Balian *et al* 1975, Fradkin *et al* 1978, Drouffe and Itzykson 1978, Itzykson 1979, Savit 1980, Dhar 1981, Melrose 1983). The reason is that, by use of the duality transformations, a high temperature region of one theory is transformed into a low temperature region of the other or, in particular cases, into the same. Hence one can obtain some information about the nature of the theory under investigation. In case of the self-duality of a model, the critical temperature becomes known. Therefore this kind of transformation can be treated as a powerful method for analysing the position of the points of phase transitions and investigating the phase structure of theories.

As fractals can simulate irregularly shaped materials (Mandelbrot 1977, Kapitulnik and Deutscher 1982) or some processes existing in nature (Witten and Sander 1981, Alexander and Orbach 1982, Gefen *et al* 1983, Given and Mandelbrot 1983, Suzuki 1983, Meakin 1983, Rammal and Toulouse 1983, Nadal *et al* 1984, Kolb and Jullien 1984, Blumen *et al* 1984) it would be interesting to study their properties and, for example, the phase structure of fractal-spin systems (Dhar 1981). This is the motivation for dealing with dual lattices and transformations for fractal systems.

Recently, discussion of a problem connected with duality for fractals has appeared (Nencka-Ficek 1985, Melrose 1986). This comment concerns the above problem.

Let us begin this comment with a systematic review of definitions and notation. We assume that the only criterion to be fulfilled while constructing the dual for a given system is conservation of its topological properties. More precisely, if X is a set of points and t a topology, then the pair $\langle X, t\rangle$ is a topological space T. Any topological space T can be characterised uniquely by a set $P=\{p_1, p_2, \ldots, p_k\}$ $(k=1, 2, \ldots)$ of its topological invariants p, i.e. its topological properties which are conserved under homeomorphisms. A dual transformation f is then a continuous one-to one function such that

$$f: T \leftrightarrow T^*$$
$$P \equiv P^*$$

where T^* is a dual topological space with P^* its set of invariants.

Let us consider an example. Let $T = \langle X, t \rangle$ be a topological space such that $X = {}^{\omega}\{0, 1\}$ is a Cantor set and t is a Tychonoff topology (Kuratowski 1968, Levy 1979). T is a perfect, compact, totally disconnected Hausdorff space. Since the set P of topological invariants is supposed to be conserved while constructing a dual to the Cantor space, then a new topological space T^* is a perfect, compact, totally disconnected Hausdorff space as well.

Before considering other dual lattices, we would like to present in a logical way some definitions which allow us to use and justify this general method of thinking. Let us introduce the definition of a graph.

Definition 1. A graph G (Massey 1984) is a topological space T consisting of a set X of one-dimensional edges $\bar{e}_{ij}$ such that the boundary $\partial\bar{e}_{ij}$ is a two-element set $\{e_i, e_j\}$, with e_i, e_j being vertices. More precisely, G is a Hausdorff space.

We can now connect the above considerations concerning topological spaces with some lattice problems. In order to do this we present the following.

Remark. Any lattice L, considered as a kind of graph G, can be viewed as a topological space T (Harary and Palmer 1973).

Definition 2. A two-dimensional lattice $L_{m,n}$ is a system of ordered pairs $\{i, j\}$ ($i = 1, \ldots, m; j = 1, \ldots, n$) of vertices e_i, e_j, such that two vertices e_i, e_j are nearest neighbours if the Euclidean distance between them is unity. Therefore the lattice $L_{m,n}$ is a direct product of two one-dimensional lattices L_m and L_n (Harary and Palmer 1973, Dhar 1977).

However, while dealing with spin system problems on lattices one mainly uses regular tilings (Kramers and Wannier 1941, Onsager 1944, Fradkin *et al* 1978, Wegner 1971, Savit 1980). A precise definition of regular tiling has been given by Coxeter.

Definition 3. Let $\{p, q\}$ be a set of p-gons such that there is a number q of the p-gons around each vertex e_i of a given system and the following relation holds:

$$\left(1 - \frac{2}{p}\right)\pi = \frac{2\pi}{q}$$

then the $\{p, q\}$ system is a regular tiling of a Euclidean plane (Coxeter 1948, 1969).

In the Euclidean case, as we can easily see, three regular tilings only exist $\{3, 6\}$, $\{4, 4\}$ and $\{6, 3\}$, i.e. triangular, square and honeycomb lattices respectively.

Considering some spin system on a lattice, there is a necessity to introduce some dual lattices and transformations (Feynman 1972, Balian *et al* 1975, Drouffe and Itzykson 1978). Coxeter's definition of duality is as follows.

Definition 4. The dual to $\{p, q\}$ is a tiling $\{q, p\}$ whose edges $\bar{e}^*_{ij}$ are perpendicular to the edges $\bar{e}_{ij}$ of $\{p, q\}$ in such a way that they cross $\bar{e}_{ij}$ in their middles (Coxeter 1948, 1969).

A similar definition was applied (Kramers and Wannier 1941, Onsager 1944) in connection with the problem of an Ising model on a square lattice.

Now we apply the above definitions to the case of the Sierpinski gasket. The Sierpinski gasket (Sierpinski 1975, Kuratowski 1968), Mandelbrot 1977) is constructed as follows. Let O be an equilateral triangle. We divide it into four congruent triangles, T_0, T_1, T_2 having their bases down, and a middle U_0 with its base up. T_0, T_1, T_2 have a common vertex with O. We call this process of partitioning O the first step of iteration. Now we divide each of the T_0, T_1, T_2 triangles into four congruent triangles in a similar way. We obtain nine triangles $T_{00}, T_{01}, \ldots, T_{22}$, with bases down and having a vertex in common with T_0, T_1 or T_2 and three triangles U_{00}, U_{01}, U_{02} having bases up. This is the second step of construction. Iterating O to infinity, one obtains the Sierpinski gasket G. The precise definition of it is

$$G = \overline{UB_{\alpha_1 \ldots \alpha_k}}$$

where

$$B_{\alpha_1 \ldots \alpha_k} = Fr(T_{\alpha_1 \ldots \alpha_k})$$

and Fr means a boundary of T.

Let us take into account the first remarks about the conservation of the topological properties of a space T, while constructing the dual one T^*. We consider the Sierpinski gasket as a topological space T with some set P of the topological properties. The Sierpinski gasket is a compact locally connected topological space (Kuratowski 1968) being a one-dimensional curve (Sierpinski 1975), the degree of deformation of which is estimated by a Hausdorff dimension $d = \ln 3/\ln 2$ (Mandelbrot 1977).

Therefore, taking the definition of the duality, we construct a compact quasi-Bethe lattice (Nencka-Ficek 1985) and a locally connected topological space with the Hausdorff dimension $d^* = \ln 3/\ln 2$.

This is one possibility of constructing the dual system to G. The other dual lattice to G (by applying the above definitions) is obtained if the dual vertices are put in the middle of every edge $\bar{e}_{ij}$ of the Sierpinski gasket. However, in the limit, this dual lattice just becomes the Sierpinski gasket.

Summarising, one should say that the only criterion for having a dual object or a dual lattice; fractal or not, is that this dual object, treated as a topological space, must have the same set of topological properties P as the original one.

The examples of some topological invariants (Kuratowski 1968, Levy 1979) are connectedness, local connectedness, compactness, being a meager set or not, and so on. Hence, if the original object is a compact one then its dual object must be compact as well. If the original object is a meagre set then its dual must be meagre as well.

Locally connected fractals characterised by a fractal dimension d, such that $1 < d \leqslant 2$ being nowhere dense continua have as the dual locally connected spaces being nowhere dense continua as well.

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